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Zeta function of the Bessel operator on the negative real axis

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Abstract. We investigate the pole structure of the zeta function $\zeta_{A_\nu}(s) = \sum_k \lambda_k^{-s}$ built out of the eigenvalues λ_k of a Bessel operator A_ν subject to Dirichlet boundary conditions at one end of the domain. This leads us to the study of ζ_{A_ν} on the negative real axis, where most of the singularities occur.

1. Introduction

The aim of this work is to provide some mathematical insight into the nature of the pole structure attached to the zeta function for the Bessel operator on the negative real axis, under specific boundary conditions. Generally speaking, the reasons why this object is of general interest are well known: for a huge category of physically motivated problems, the Hamiltonian is interwoven with the operator of the Bessel equation; e.g. classical vibrating strings and drumheads, heat conduction in cylinders, normal modes in resonant cavities, Fraunhofer diffraction through circular apertures, quantum free particles in cylindrical or spherical domains under special boundary conditions, and the Casimir effect for perfectly conducting shells with the same type of symmetry [1].

As for quantum particles, the two-dimensional case has been further complicated in two different ways: altering the shape of the boundary (quantum billiards [2, 3]) and threading the domain with a magnetic flux line (Aharonov–Bohm quantum billiards [3, 4]). The extension of the latter to spherical domains has been considered by the authors in [5]. A consequence of these studies is the need to know the zeros $\{j_{\nu n}\}$ of the Bessel function J_ν for arbitrary real ν. The method put forward in [3, 4], and further developed in [5], is based on their numerical evaluation from the zeta function

$$\zeta_\nu(s) = \sum_{n=1}^{\infty} j_{\nu n}^{-s} \qquad \mathrm{Re}\, s > 1. \tag{1.1}$$

In particular, one takes advantage of the properties already noticed by Euler [6], which lead to

$$\lim_{k\to\infty} [\zeta_\nu(2k)]^{-1/(2k)} = \lim_{k\to\infty} \left[\frac{\zeta_\nu(2k)}{\zeta_\nu(2k+2)}\right]^{1/2} = j_{\nu 1}. \tag{1.2}$$

Once $j_{\nu 1}$ has been found, we can delete the corresponding term from $\zeta_\nu(2k)$ and apply the same procedure to the resulting sum, thus obtaining $j_{\nu 2}$, etc.

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After studying ζ_ν in some detail, it was shown that its values at positive even integers satisfy two recursive laws, namely the 'quadratic' law

$$\begin{cases} \zeta_\nu(2) = \dfrac{1}{4(\nu+1)} \\ \zeta_\nu(2n) = \dfrac{1}{n+\nu} \displaystyle\sum_{l=1}^{n-1} \zeta_\nu(2l)\zeta_\nu(2n-2l) \qquad l \geqslant 2 \end{cases} \tag{1.3}$$

and the 'linear' law

$$\frac{1}{4n!(n+\nu+1)!} = \sum_{k=0}^{n} \frac{(-1)^k 2^{2k}}{(n-k)!(n-k+\nu)!} \zeta_\nu(2k+2) \qquad n \geqslant 0. \tag{1.4}$$

Either of these allows a quick calculation of the values of $\zeta_\nu(2n)$, the first of which are quoted in the appendix. Moreover, (1.3) and (1.4) have proven to be suitable for the calculation of $j_{\nu 1}$ by means of programming with recursive procedures or functions.

While most of those findings concern the values of ζ_ν for positive integers, little is known about this function on the negative part of the real axis. In this paper we shall deal with this portion of the domain. Apart from possible applications (e.g. along the lines suggested in [7]), the authors share the view that the mathematical content of this subject can be interesting enough by itself.

Section 2 is a survey of the heat kernel series method. Its application to the Bessel operator in one dimension is described in section 3, where two different recursive laws for the coefficients are found. The residues of ζ_ν at negative odd integers and its finite values at non-positive integers are also calculated. In section 4, we provide an independent derivation of the same results by analytic continuation based on complex-plane integration techniques, without resorting to heat kernel formalism. A note on the extension of that method to the case of standard homogeneous boundary conditions is also given. Some final comments appear in section 5.

2. Asymptotic expansion of the heat kernel

The zeta function for an operator A, of positive eigenvalues $\{\lambda_k\}$, is defined to be

$$\zeta_A(s) = \sum_k \lambda_k^{-s}. \tag{2.1}$$

If A has zero eigenvalues, they are omitted from this sum and dealt with separately when this is of interest. As a rule, (2.1) makes sense only for $\mathrm{Re}\, s$ larger than some positive s_0, which is the rightmost real pole of the function in question. By means of the integral representation

$$\zeta_A(s) = \frac{1}{\Gamma(s)} \int_0^\infty \mathrm{d}t\, t^{s-1}\, \mathrm{Tr}\, \mathrm{e}^{-tA} \tag{2.2}$$

where $\mathrm{Tr}\, \mathrm{e}^{-tA} = \sum_k \mathrm{e}^{-t\lambda_k}$, $\zeta_A(s)$ can be analytically continued to other values of s. A way of doing so is to take advantage of the heat kernel expansion [8]

$$\mathrm{Tr}\, \mathrm{e}^{-tA} \sim \frac{1}{(4\pi t)^{D/2}} \sum_{n=0}^{\infty} C_n t^{n/2} \tag{2.3}$$

which is *asymptotic* for small t (hence the $\sim$ sign). D is the space dimension and C_n are calculable coefficients independent of t. The behaviour for large t is

$$\mathrm{Tr}\, \mathrm{e}^{-tA} \sim \mathrm{e}^{-t\varepsilon} \qquad t \to \infty \tag{2.4}$$

where $\varepsilon > 0$ denotes the lowest eigenvalue. After splitting the integration domain into $[0, 1]$ and $[1, \infty)$ and substituting (2.3) in the first part of the integral,

$$\zeta_A(s) \sim \frac{1}{(4\pi)^{D/2}\Gamma(s)}\left[\sum_{n=0}^{\infty} C_n \frac{1}{s+(n-D)/2} + f(s)\right] \tag{2.5}$$

where by virtue of (2.4) f is analytic for any s. Let us look for possible poles, which can only be at points of the form $s = (D-n)/2$, $n = 0, 1, 2, \ldots$:

(i) $n < D$

There are poles of order one at $s = \frac{D}{2}, \frac{D-1}{2}, \ldots, \frac{1}{2}$ with residues

$$\mathrm{Res}\left[\zeta_A(s), s = \frac{D-n}{2}\right] = \frac{1}{(4\pi)^{D/2}\Gamma\left(\frac{D-n}{2}\right)} C_n, \quad n = 0, 1, \ldots, D-1. \tag{2.6}$$

(ii) $n \geqslant D$

(a) $$s = \frac{D-n}{2} = -m \qquad m \in N$$

($s = 0$ is included here). Given that $\lim_{s\to -m} \frac{1}{\Gamma(s)(s+m)} = (-1)^m m!$ is finite, ζ_A has no singularity at these points. In fact, its value is

$$\zeta_A(-m) = \frac{1}{(4\pi)^{D/2}}(-1)^m m!\, C_{D+2m}. \tag{2.7}$$

(b) $$s = \frac{D-n}{2} = -\left(m+\tfrac{1}{2}\right) \qquad m \in N$$

All of them are potential poles of order one, with residues

$$\mathrm{Res}\left[\zeta_A(s), s = -\left(m+\tfrac{1}{2}\right)\right] = \frac{1}{(4\pi)^{D/2}\Gamma\left(-\left(m+\frac{1}{2}\right)\right)} C_{D+2m+1} \qquad m \in N. \tag{2.8}$$

Of course, they become true singularities wherever these residues are non-zero.

We will focus this study on the Bessel operator in $D = 1$

$$A = A_\nu \equiv -\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r} r \frac{\mathrm{d}}{\mathrm{d}r} + \frac{\nu^2}{r^2} \tag{2.9}$$

with eigenfunctions subject to the following boundary conditions: they must be regular at $r = 0$ and vanish at $r = 1$. These solutions are therefore of the form $J_\nu(j_{\nu n} r)$, where $j_{\nu n}$ is the nth positive zero of J_ν. As a result, the eigenvalues of A_ν are given by $j_{\nu n}^2$. Thus

$$\zeta_{A_\nu}(z) = \sum_{n=1}^{\infty} j_{\nu n}^{-2z} \qquad \mathrm{Re}\, z > \tfrac{1}{2}. \tag{2.10}$$

For convenience, we will rather handle the zeta function (1.1), i.e. $\zeta_\nu(s) = \zeta_{A_\nu}(s/2)$. The bound $\mathrm{Re}\, s > 1$ in its series definition comes from our knowledge that the rightmost real pole is now at $s/2 = D/2$, i.e. $s = 1$ (or from considering the asymptotic growth of $j_{\nu n}$ and comparing with the Hurwitz zeta function).

Particularly easy cases are $\nu = \pm\frac{1}{2}$, because

$$J_{\pm 1/2}(z) = \sqrt{\frac{2}{\pi z}} \begin{cases} \sin z \\ \cos z \end{cases}$$

and

$$j_{\pm 1/2\; n} = \begin{cases} \pi n \\ \pi\left(n+\frac{1}{2}\right). \end{cases}$$

Then

$$\zeta_{\pm 1/2}(s) = \pi^{-s} \begin{cases} \zeta(s) \\ \zeta(s, \frac{1}{2}) \end{cases} \tag{2.11}$$

$\zeta(s)$ and $\zeta(s, a)$ being the Riemann and Hurwitz zeta functions. They have only one pole at $s = 1$ with unit residue. Therefore, the residues for the negative poles of ζ_ν must vanish at $\nu = \pm\frac{1}{2}$. The same should happen to its finite values at negative even integers, since $\zeta(-2m) = \zeta\left(-2m, \frac{1}{2}\right) = 0$, $m = 1, 2, \ldots$.

When considering $\zeta_\nu(s)$, the negative poles are located at $s = -(2m + 1), m \in N$. From (2.8) for $D = 1$, taking into account the argument rescaling from ζ_{A_ν} to ζ_ν and using $\Gamma(z)\Gamma(1 - z) = \pi \csc \pi z$, we can put

$$\mathrm{Res}_{-(2m+1)} \equiv \mathrm{Res}[\zeta_\nu(s), s = -(2m + 1)] = \frac{(-1)^{m+1}}{\pi} c_{2m+2} \tag{2.12}$$

where we have introduced the notation

$$c_n \equiv \frac{1}{\sqrt{\pi}} C_n \Gamma\left(\frac{n+1}{2}\right) \qquad n \in N. \tag{2.13}$$

Concerning the finite values at non-positive even integers, they are easily read from $\zeta_\nu(-2m) = \zeta_{A_\nu}(-m)$ and (2.7), which for $D = 1$ give

$$\zeta_\nu(-2m) = \tfrac{1}{2}(-1)^m c_{2m+1} \qquad m = 0, 1, 2, \ldots . \tag{2.14}$$

As we have just shown, the c_n's encode the necessary information to describe the pole structure. Our aim is to find exact expressions to determine them up to any arbitrary n.

3. Coefficients of the heat kernel expansion for the Bessel operator

First, we will apply Moss' method [9]. Instead of the operator A, consider $A + x^2$ for sufficiently large x:

$$\begin{aligned} \zeta_{A+x^2}(s) &= \sum_k (\lambda_k + x^2)^{-s} \\ &= \frac{1}{\Gamma(s)} \int_0^\infty \mathrm{d}t\, t^{s-1} \mathrm{e}^{-tx^2}\, \mathrm{Tr}\, \mathrm{e}^{-tA}. \end{aligned} \tag{3.1}$$

Making use of (2.3), taking advantage of Watson's lemma and integrating term by term—this time without splitting the domain—we get

$$\zeta_{A+x^2}(s) \sim \frac{1}{\Gamma(s)(4\pi)^{D/2}} \sum_{n=0}^{\infty} C_n \Gamma\left(s + \frac{n-D}{2}\right) x^{-2s-n+D}. \tag{3.2}$$

In particular, for $D = 1$, $s = 1$ —which is one of the cases not considered in that work— and using (2.13), this asymptotic equality turns into

$$\zeta_{A+x^2}(1) \sim \tfrac{1}{2} \sum_{n=0}^{\infty} c_n x^{-(n+1)}. \tag{3.3}$$

All this is valid for any operator A such that these expressions make mathematical sense. From now on, we specify them to the Bessel operator in the above-quoted conditions. We begin with the expression of J_ν as an infinite product ([10], vol 2, p 61)

$$J_\nu(z) = \frac{z^\nu}{2^\nu \Gamma(\nu + 1)} \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{j_{\nu n}^2}\right). \tag{3.4}$$

After the change $z = ix$, we get the modified Bessel functions $I_\nu(x) = \mathrm{i}^{-\nu} J_\nu(ix)$. Taking the logarithmic derivative of (3.4) and using the property

$$\frac{\partial}{\partial x}(x^{-\nu} I_\nu(x)) = x^{-\nu} I_{\nu+1}(x) \tag{3.5}$$

we arrive at

$$\frac{I_{\nu+1}(x)}{I_\nu(x)} = 2x \sum_{n=1}^{\infty} \frac{1}{j_{\nu n}^2 + x^2} = 2x\ \zeta_{A_\nu + x^2}(1). \tag{3.6}$$

The outcome so far is not new, in the sense that the above relation is essentially the same as equation (A.5) in [4]. However, while in this reference the author employs methods involving continued fractions or, alternatively, power-series expansions, here we make use of *asymptotic* identities.

Comparing (3.6) and (3.3), we obtain

$$\frac{I_{\nu+1}(x)}{I_\nu(x)} \sim \sum_{n=0}^{\infty} c_n x^{-n}. \tag{3.7}$$

This is our starting point. From this we will get two different recursions for the c_n's and thus calculate the residues.

Lemma 1. The c_n's obey the 'quadratic' recursive law

$$c_0 = 1 \tag{3.8}$$

$$c_1 = -\left(\nu + \tfrac{1}{2}\right) \tag{3.9}$$

$$c_2 = c_3 = \tfrac{1}{2}\left(\nu^2 - \tfrac{1}{4}\right) \tag{3.10}$$

$$2c_n = (n-1)c_{n-1} - \sum_{l=2}^{n-2} c_l c_{n-l} \qquad n \geqslant 4. \tag{3.11}$$

Proof. Differentiating (3.7), and with the help of (3.5) and of $\frac{\partial}{\partial x}\left(x^{\nu+1} I_{\nu+1}(x)\right) = x^{\nu+1} I_\nu(x)$, we come to

$$1 - \frac{2\nu+1}{x}\frac{I_{\nu+1}(x)}{I_\nu(x)} - \left(\frac{I_{\nu+1}(x)}{I_\nu(x)}\right)^2 \sim -\sum_{n=0}^{\infty} c_n\ n x^{-(n+1)}. \tag{3.12}$$

Replacing again $I_{\nu+1}(x)/I_\nu(x)$ with the right-hand side of (3.7), and after some index rearrangements, the ensuing relation reads

$$1 - c_0^2 + \sum_{n=1}^{\infty}\left[(n - 2\nu - 2)c_{n-1} - \sum_{l=0}^{n} c_l c_{n-l}\right] x^{-n} \sim 0. \tag{3.13}$$

Since it has to hold for any possible x, the identity must be satisfied separately for each power of x. The terms in x^0 yield $c_0^2 = 1$. Given that $c_0 = \lim_{x\to\infty} I_{\nu+1}(x)/I_\nu(x)$, the plus sign must be chosen, i.e. we get (3.8) and, in consequence, $C_0 = 1$.

As for the coefficients in x^{-n}, $n \geqslant 1$, we have $(n - 2\nu - 2)c_{n-1} = \sum_{l=0}^{n} c_l c_{n-l}$, $n \geqslant 1$. Solving recurrently from (3.8), we find (3.9), (3.10). Substituting the values of c_0 and c_1, we write a more convenient form of the above recursion, namely (3.11). □

The calculated values of $c_4, c_5, \ldots$, have been listed in the appendix (formula (A.2)). $c_0 = 1$ originates from the residue for the pole of $\zeta_\nu(s)$ at $s = 1$, which is always present. All the other c_n's vanish for $\nu = +\frac{1}{2}$ from $n = 2$ on, and for $\nu = -\frac{1}{2}$ from $n = 1$ on. This slight difference stems from the fact that, by virtue of (2.14), c_1 is twice the finite value of the zeta function at $s = 0$; the Hurwitz zeta function vanishes at $s = 0$ when the second argument equals $\frac{1}{2}$, while the Riemann zeta function does not vanish at $s = 0$† , as a result of which

$$c_1\left(\nu = \tfrac{1}{2}\right) = 2\zeta_{1/2}(0) = 2\zeta(0) = -1$$
$$c_1\left(\nu = -\tfrac{1}{2}\right) = 2\zeta_{-1/2}(0) = 2\zeta\left(0, \tfrac{1}{2}\right) = 0.$$

Apart from $\nu = -\frac{1}{2}\left(+\frac{1}{2}\right)$ for $n \geqslant 1(2)$, there are other roots for particular c_n's, all of them real‡, forming a somewhat more involved pattern. Since the c_n's are even polynomials in ν, we have studied their zeros referred to the variable ν^2 rather than ν. Examination of table 1 shows that the kth root takes on values which are 'around' $\frac{1}{4}(2k+1)^2$ ($k = 0, 1, 2\ldots$). However, while the one for $k = 0$ always shows up exactly as $\frac{1}{4}$, and the one for $k = 1$ seems to decrease monotonically to the value $\frac{9}{4}$ for increasing n—it has not yet actually arrived at this value when already $n = 16$—for larger k's there appears to be a damped oscillation about $\frac{1}{4}(2k+1)^2$, which, in the case of $k = 2$, coincides with $\frac{25}{4}$ for $n = 10$ and $n = 16$, without remaining there for successive n's. By (2.12), this fact entails the *finiteness* of $\zeta_{\pm 5/2}(s)$ at $s = -9$ and $s = -15$ (in addition to $s = -3$, as signalled by the $\frac{25}{4}$ for $n = 4, k = 1$), while maintaining poles at all the intervening negative integers. Moreover, we have algebraically computed further c_n's up to $n = 22$ (which we have refrained from listing in (A.2) any further). The pair of zeros $\nu^2 = \frac{25}{4}$ shows up again only at $n = 22$, telling us that $\zeta_{\pm 5/2}(s)$ is finite at $s = -21$ but has poles at the negative integers between this point and $s = -15$. Thus, we feel that we have grounds now to believe that this function will cease to be singular at all the negative integers of the form $s = -(3+6m)$, $m = 0, 1, 2, \ldots$.

3.1. Linear recursion

Lemma 2. The c_n's satisfy the 'linear' recursive rule

$$\sum_{m=0}^{n}(-1)^m 2^m \frac{n!}{(n-m)!}\frac{\Gamma\left(\nu+n-m+\frac{1}{2}\right)}{\Gamma\left(\nu-n+m+\frac{1}{2}\right)}c_m = \frac{\Gamma\left(\nu+n+\frac{3}{2}\right)}{\Gamma\left(\nu-n+\frac{3}{2}\right)} \qquad n = 0, 1, 2, \ldots. \tag{3.14}$$

Proof. We recast (3.7) as

$$I_{\nu+1}(x) \sim I_\nu(x)\sum_{n=0}^{\infty} c_n x^{-n}. \tag{3.15}$$

Given that the heat kernel expansion on the RHS is asymptotic for large x, it is sensible to employ a similar one for the modified Bessel functions, that is

$$I_\nu(x) \sim \frac{e^x}{\sqrt{2\pi x}}\sum_{n=0}^{\infty} a_{\nu n} x^{-n} \tag{3.16}$$

with

$$a_{\nu n} = \frac{(-1)^n}{n!8^n}\prod_{k=1}^{n}\left[4\nu^2 - (2k-1)^2\right] = \frac{(-1)^n}{n!2^n}\frac{\Gamma\left(\nu+n+\frac{1}{2}\right)}{\Gamma\left(\nu-n+\frac{1}{2}\right)}. \tag{3.17}$$

Substituting the series in both the LHS and the RHS of (3.15),

$$\sum_{n=0}^{\infty}\left[a_{\nu+1\ n} - \sum_{m=0}^{n} a_{\nu\ n-m} c_m\right] x^{-n} \sim 0. \tag{3.18}$$

† In fact, one has $\zeta(0) = -\frac{1}{2}$, $\zeta(0, a) = \frac{1}{2} - a$ (see e.g. [10]).

‡ Here, the Descartes theorem allows one to anticipate that all roots shall be real.

Table 1. Squares of the zeros of c_n, labelled by $k = 0, 1, 2, \ldots$, for increasing n. $\frac{1}{4}$ is present, for $k = 0$, at any n. At $k = 1$ we notice an approach from above to $\frac{9}{4}$. For $k \geqslant 2$ we observe 'oscillations', with decreasing amplitude, around $\frac{1}{4}(2k+1)^2$, coinciding with this exact figure on some occasions (see $k = 2$ for $n = 10$ and $n = 16$). (Columns $k = 1$ and $k = 2$ had to be listed at higher precision.)

	Values of ν^2 such that $c_n(\nu) = 0$							
	k							
n	0	1	2	3	4	5	6	7
2	$\frac{1}{4}$							
3	$\frac{1}{4}$							
4	$\frac{1}{4}$	$\frac{25}{4}$						
5	$\frac{1}{4}$	$\frac{13}{4}$						
6	$\frac{1}{4}$	2.58809621	25.911904					
7	$\frac{1}{4}$	2.36581874	10.884181					
8	$\frac{1}{4}$	2.28646215	7.690693	66.7728				
9	$\frac{1}{4}$	2.26012898	6.613306	24.8766				
10	$\frac{1}{4}$	2.25246126	$\frac{25}{4}$	16.0951	136.4024			
11	$\frac{1}{4}$	2.25052767	6.176709	13.0229	47.0498			
12	$\frac{1}{4}$	2.25010121	6.200318	11.9241	28.5022	242.3732		
13	$\frac{1}{4}$	2.25001759	6.230507	11.7512	21.7723	79.2460		
14	$\frac{1}{4}$	2.25000280	6.244948	11.9816	19.0794	45.6863	392.2578	
15	$\frac{1}{4}$	2.25000041	6.249211	12.2175	18.5321	33.1919	123.3092	
16	$\frac{1}{4}$	2.25000006	$\frac{25}{4}$	12.2891	19.3694	27.5242	68.4384	593.6290
17	$\frac{1}{4}$	2.25000001	6.250045	12.2765	20.6196	25.5550	47.7159	181.0830
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
	Values of $\frac{1}{4}(2k+1)^2$, $k = 0, 1, 2, \ldots$							
	0.25	2.25	6.25	12.25	20.25	30.25	42.25	56.25

The validity of any x requires $a_{\nu+1\ n} = \sum_{m=0}^{n} a_{\nu\ n-m} c_m$, $n \geqslant 0$, which, by (3.17), yields (3.14). □

By taking $n = 0$ in equation (3.14), the equality (3.8) follows immediately. For successive values of n we again obtain the identities (3.9), and (3.10), as well as those in (A.2).

Although work with linear recursion is usually easier than with quadratic recursion, in the authors' view the simplicity of the coefficients in (3.11), as compared to those in (3.14), makes the quadratic rule actually more amenable to algebraic computation.

3.2. *Explicit values of the residues and of $\zeta_\nu(2m)$*

Recalling the result for the only positive pole at $s = 1$ (2.12) and the coefficients (3.8), (3.9), (3.10), (A.2), we find the precise form for the residues of the poles of ζ_ν. To illustrate this, we write down those for the rightmost five poles:

$$
\begin{aligned}
\mathrm{Res}_1 &= \frac{1}{\pi} \\
\mathrm{Res}_{-1} &= -\frac{1}{2\pi}\left(\nu^2-\frac{1}{4}\right) \\
\mathrm{Res}_{-3} &= -\frac{1}{8\pi}\left(\nu^2-\frac{25}{4}\right)\left(\nu^2-\frac{1}{4}\right) \\
\mathrm{Res}_{-5} &= -\frac{1}{16\pi}\left(\nu^4-\frac{57}{2}\nu^2+\frac{1073}{16}\right)\left(\nu^2-\frac{1}{4}\right) \\
\mathrm{Res}_{-7} &= -\frac{45}{512\pi}\left(\nu^6-\frac{307}{74}\nu^4+\frac{54703}{80}\nu^2-\frac{375733}{320}\right)\left(\nu^2-\frac{1}{4}\right). \\
&\vdots
\end{aligned}
\tag{3.19}
$$

Except for the first residue, all of them vanish when $\nu = \frac{1}{2}$, as expected.

Next, we turn to the finite values of $\zeta_\nu(-2m)$. Equations (2.14) and (3.8), (3.9), (3.10), (A.2) give these quantities up to any desired m. As an example, we list the first five, which are

$$
\begin{aligned}
\zeta_\nu(0) &= -\frac{1}{2}\left(\nu+\frac{1}{2}\right) \\
\zeta_\nu(-2) &= -\frac{1}{4}\left(\nu^2-\frac{1}{4}\right) \\
\zeta_\nu(-4) &= -\frac{1}{4}\left(\nu^2-\frac{13}{4}\right)\left(\nu^2-\frac{1}{4}\right) \\
\zeta_\nu(-6) &= -\frac{1}{4}\left(\nu^4-\frac{53}{4}\nu^2+\frac{103}{4}\right)\left(\nu^2-\frac{1}{4}\right) \\
\zeta_\nu(-8) &= -\frac{1}{4}\left(\nu^6-\frac{135}{4}\nu^4+\frac{3771}{16}\nu^2-\frac{23797}{64}\right)\left(\nu^2-\frac{1}{4}\right) \\
&\vdots
\end{aligned}
\tag{3.20}
$$

By virtue of (2.11) we can now get back the known results for the Riemann and Hurwitz zeta functions $\zeta(0) = -\frac{1}{2}$, $\zeta\left(0, \frac{1}{2}\right) = 0$, and $\zeta(-2n) = \zeta\left(-2n, \frac{1}{2}\right) = 0$, $n = 1, 2, \ldots$,† after just setting $\nu = \pm\frac{1}{2}$. This shows the 'explosive' nature of ζ_ν's, for $\nu \neq \pm\frac{1}{2}$—or other roots of the c_n's—as compared with the Riemann or Hurwitz ζ. Actually, not only can the former be non-zero where the latter vanish, but they also blow up at some points where the Riemann and Hurwitz functions are finite. Cases like $\nu = \pm\frac{5}{2}$ are in a sense closer to these classical functions, as the good behaviour at *some* negative integers is recovered.

The value of $\zeta_\nu(0)$ in (3.20) was already conjectured in [4] (equation (A.17)), while in the present work it has been shown to be correct.

† The vanishing of the Hurwitz function at these points follows from $\zeta(-m, x) = -\frac{B_{m+1}(x)}{m+1}$, $B_m(x)$ being the mth Bernoulli polynomial, and from the property $B_m(1-x) = (-1)^m B_m(x)$, which, in turn, implies $B_{2n+1}\left(\frac{1}{2}\right) = 0$.

4. Derivation by complex integration

4.1. Alternative proof of lemma 1

The zeta function (1.1) can be expressed as the complex-plane integral

$$\zeta_\nu(z) = -\frac{1}{2\pi \mathrm{i}}\int_{C_\epsilon} \mathrm{d}t\,\frac{J'_\nu(t)}{J_\nu(t)}\,t^{-z} = \frac{\mathrm{i}z}{2\pi}\int_{C_\epsilon}\mathrm{d}t\,\ln\left[J_\nu(t)\right]t^{-z-1} \qquad \mathrm{Re}\,z > 1 \tag{4.1}$$

where C_ϵ is a sector-like contour between two arcs of circumference—of radii ϵ and R—centred at the origin and two radial lines connecting them. To be more precise, $\epsilon \leqslant |t| \leqslant R$, $0 < \epsilon < j_{\nu 1}$, $R \to \infty$, and $|\arg(t)| \leqslant \theta$, $0 < \theta \leqslant \pi/2$. We will eventually take $\theta = \pi/2$. It is immediately seen that, inside this region, the equation $J_\nu(j_{\nu n}) = 0$ has only real solutions. Let $C_{+(\epsilon)}$ and $C_{-(\epsilon)}$ be the upper and lower halves, respectively, of this contour. The behaviour of J_ν along $C_{\pm(\epsilon)}$ is given by

$$J_\nu(t) = \frac{1}{\sqrt{2\pi t}}\mathrm{e}^{\mp\mathrm{i}t}\mathrm{e}^{\pm\mathrm{i}(\nu+1/2)\pi/2}\left[1+\mathrm{O}(1/t)\right]. \tag{4.2}$$

Therefore

$$\zeta_{\nu\epsilon}^{(\pm)}(z) = \frac{\mathrm{i}z}{2\pi}\left\{\int_{C_{\pm(\epsilon)}}\mathrm{d}t\,t^{-z-1}\ln\left[\sqrt{2\pi t}\;\mathrm{e}^{\pm\mathrm{i}[t-(\nu+1/2)\pi/2]}\;J_\nu(t)\right]\right.$$
$$\left.-\int_{C_{\pm(\epsilon)}}\mathrm{d}t\,t^{-z-1}\ln\left[\sqrt{2\pi t}\;\mathrm{e}^{\pm\mathrm{i}[t-(\nu+1/2)\pi/2]}\right]\right\}. \tag{4.3}$$

Regarded as a function of z, the first integral is analytic for $\mathrm{Re}\,z > -1$, and the second only for $\mathrm{Re}\,z > 1$, which is where ζ_ν can be represented by the series (1.1). Thus we have $\zeta_{\nu\epsilon}^{(\pm)}(z)$ defined for $\mathrm{Re}\,z > 1$. Next, we do the sum $\zeta_\nu(z) = \zeta_{\nu\epsilon}^{(+)}(z) + \zeta_{\nu\epsilon}^{(-)}(z)$, performing the addition of the respective second terms, and integrating their result for $\theta = \pi/2$. The outcome is

$$\zeta_\nu(z) = \frac{\mathrm{i}z}{2\pi}\sum_{(\pm)}\int_{C_{\pm(\epsilon)}}\mathrm{d}t\,t^{-z-1}\ln\left[\sqrt{2\pi t}\mathrm{e}^{\pm\mathrm{i}[t-(\nu+1/2)\pi/2]}J_\nu(t)\right]$$
$$+\frac{\mathrm{i}z}{\pi}\left[\frac{\mathrm{i}\epsilon^{1-z}}{1-z}+\mathrm{i}\left(\nu+\frac{1}{2}\right)\frac{\pi}{2z}\epsilon^{-z}\right] \tag{4.4}$$

valid for $\mathrm{Re}\,z > 1$. However, we now observe that the second term explicitly provides its own analytic continuation (it is actually a meromorphic function), while the first is, as already remarked, analytic for $\mathrm{Re}\,z > -1$. We thus see that ζ_ν has a simple pole at $z = 1$ with residue $1/\pi$. Moreover, it is also plain that $\zeta_\nu(0) = -\frac{1}{2}\left(\nu+\frac{1}{2}\right)$.

Let us now consider the restriction of (4.4) to the strip $-1 < \mathrm{Re}\,z < 0$, keeping $\theta = \pi/2$ fixed. The limit $\epsilon \to 0$ yields the fundamental expression

$$\zeta_\nu(z) = \frac{z}{\pi}\sin\left(\frac{\pi}{2}z\right)\int_0^\infty \mathrm{d}\rho\,\rho^{-z-1}\ln\left[\sqrt{2\pi\rho}\,\mathrm{e}^{-\rho}I_\nu(\rho)\right] \qquad -1 < \mathrm{Re}\ z < 0. \tag{4.5}$$

It has now to be continued—in the range of $\mathrm{Re}\,z$—to the left of the real axis. Let

$$G(\rho) \equiv \ln\left[\sqrt{2\pi\rho}\,\mathrm{e}^{-\rho}I_\nu(\rho)\right]. \tag{4.6}$$

This function admits an asymptotic expansion of the form $G(\rho) \sim \sum_{n=1}^{\infty} a_n\rho^{-n}$, for $\rho \to \infty$, which can be found term by term. On the other hand, it can be proved that G satisfies

$$\rho^2 G'' + \rho^2 G'^2 + 2\rho^2 G' + \tfrac{1}{4} - \nu^2 = 0. \tag{4.7}$$

After substituting the asymptotic series of $G(\rho)$ in this differential equation, we obtain

$$\begin{aligned} a_1 &= \tfrac{1}{2}\left(\tfrac{1}{4}-\nu^2\right) \\ a_2 &= \tfrac{1}{4}\left(\tfrac{1}{4}-\nu^2\right) \\ 2(n+1)a_{n+1} &= n(n+1)a_n + \sum_{k=2}^{n}(k-1)(n-k+1)a_{k-1}a_{n-k+1} \qquad n \geqslant 2. \end{aligned} \tag{4.8}$$

Deleting and adding to the $G(\rho)$ in the integrand a term $\Theta(\rho-1)\sum_{k=1}^{p} a_k\rho^{-k}$ (where $\Theta(\rho-1)$ denotes the unit step function switching on at $\rho = 1$), and performing the integration of the last piece, one comes to

$$\begin{aligned} \zeta_\nu(z) &= \frac{z}{\pi}\sin\left(\frac{\pi z}{2}\right)\int_0^\infty d\rho\,\rho^{-z-1}\left[G(\rho) - \Theta(\rho-1)\sum_{k=1}^{p} a_k\rho^{-k}\right] \\ &\quad + \frac{z}{\pi}\sin\left(\frac{\pi z}{2}\right)\sum_{k=1}^{p}\frac{a_k}{(z+k)}. \end{aligned} \tag{4.9}$$

This holds, in principle, for $-1 < \operatorname{Re} z < 0$, but it gives an explicit analytic continuation to $-p-1 < \operatorname{Re} z < 0$, from which we find

$$\begin{aligned} &\zeta_\nu(-2l) = -(-1)^l\, l\, a_{2l} && l \in \boldsymbol{N}^* \\ &\operatorname{Res}[\zeta_\nu(z), z = -(2l+1)] = \frac{(-1)^l}{\pi}(2l+1)a_{2l+1} && l \in \boldsymbol{N}. \end{aligned} \tag{4.10}$$

After comparing them with (2.12) and (2.14), we realize that (4.8) is identical, up to the notational change $c_n = -(n-1)a_n$, to the quadratic recurrence (3.8)–(3.11).

4.2. Extension to standard homogeneous boundary conditions

We are taking the same operator A_ν as in (2.9) but defined now on functions ϕ regular at $r = 0$ and such that $a\phi(1) + b\phi'(1) = 0$, where $a, b \in \boldsymbol{R}$ and $|a| + |b| > 0$, which ensures hermiticity. Furthermore, the requirement $\operatorname{sign}(a) = \operatorname{sign}(b)$ will be enough to guarantee that A_ν is positive as well, and we shall therefore denote its eigenvalues by λ^2, with $\lambda \in \boldsymbol{R}^+$. The study of ζ_{A_ν} can go through by using $\zeta_{\theta,\nu}(z) \equiv \sum_{n=1}^{\infty}[\lambda_n(\theta,\nu)^z]^{-1}$, where $\cos\theta\; J_\nu(\lambda_n(\theta,\nu)) + \sin\theta\;\lambda_n(\theta,\nu)\; J_\nu'(\lambda_n(\theta,\nu)) = 0,\; 0 \leqslant \theta < \pi$. This zeta function is now expressed as the complex-plane integral

$$\zeta_{\theta,\nu}(z) = \frac{iz}{2\pi}\int_{C_\epsilon} dt\; \ln\left[\cos\theta\; J_\nu(t) + \sin\theta\; t\; J_\nu'(t)\right]\; t^{-z-1} \tag{4.11}$$

where C_ϵ is the same circuit as in the previous subsection, and will be decomposed into the same pieces $C_{+(\epsilon)}$ and $C_{-(\epsilon)}$.

We will consider $\theta \neq 0$ (the $\theta = 0$ case has already been studied). In order to ensure the positivity of A_ν and, thus, that there is no λ on the imaginary axis, we assume this analysis is restricted to $0 < \theta \leqslant \pi/2$. It is easy to see that, along $C_{\pm(\epsilon)}$,

$$\begin{aligned} &\cos\theta\; J_\nu(t) + \sin\theta\; t\; J_\nu'(t) \\ &\quad = \mp\sqrt{\frac{t}{2\pi}}\, e^{\mp i[t-(\nu+1/2)\pi/2]}\sin\theta\left[1 \mp \frac{i}{8t}(4\nu^2+3) \pm i\frac{\tan^{-1}\theta}{t} + O\left(\frac{1}{t^2}\right)\right]. \end{aligned}$$

Therefore, we put

$$\zeta_{\theta,\nu}(z) = \sum_{(\pm)} \zeta_{\theta,\nu,\epsilon}^{(\pm)}(z)$$
$$= \frac{\mathrm{i}z}{2\pi} \sum_{(\pm)} \int_{C_{\pm(\epsilon)}} \mathrm{d}t \ \ln\left[\pm\mathrm{i}\sqrt{\frac{2\pi}{t}}\mathrm{e}^{\mp\mathrm{i}[t-(\nu+1/2)\pi/2]}\sec^{-1}\theta[\cos\theta \ J_\nu(t) + \sin\theta \ t \ J'_\nu(t)]\right]$$
$$- \frac{\mathrm{i}z}{2\pi} \sum_{(\pm)} \int_{C_{\pm(\epsilon)}} \mathrm{d}t \ln\left[\pm\mathrm{i}\sqrt{\frac{2\pi}{t}}\mathrm{e}^{\mp\mathrm{i}[t-(\nu+1/2)\pi/2]}\sec^{-1}\theta\right]. \tag{4.12}$$

This holds for $\mathrm{Re}\, z > 1$, but the first term gives an analytic function for $\mathrm{Re}\, z > 1$, and the second can be straightforwardly calculated:

$$\frac{\mathrm{i}z}{2\pi} \sum_{(\pm)} \int_{C_{\pm(\epsilon)}} \mathrm{d}t \ln\left[\pm\mathrm{i}\sqrt{\frac{2\pi}{t}}\mathrm{e}^{\mp\mathrm{i}[t-(\nu+1/2)\pi/2]}\sec^{-1}\theta\right] = \frac{\mathrm{i}z}{2\pi}\left[\frac{\mathrm{i}\epsilon^{1-z}}{1-z} + \mathrm{i}\left(\nu - \frac{1}{2}\right)\frac{\pi}{2z}\epsilon^{-z}\right]. \tag{4.13}$$

We thus obtain an analytic extension of $\zeta_{\theta,\nu}(z)$ for $\mathrm{Re}\, z > 1$, with a pole at $z = 1$ of residue $1/\pi$. Also $\zeta_{\theta,\nu}(0) = -\frac{1}{2}(\nu - \frac{1}{2})$. Considering $-1 < \mathrm{Re}\, z < 0$, and taking $\epsilon \to 0^+$, we have

$$\zeta_{\theta,\nu}(z) = \sum_{(\pm)} \frac{\mathrm{i}z}{2\pi}(\pm 1)\mathrm{e}^{\mp\mathrm{i}z\pi/2} \int_0^\infty \mathrm{d}\rho \ \rho^{-z-1}$$
$$\times \ln\left[(\pm\mathrm{i})\sqrt{\frac{2\pi}{\rho}}\mathrm{e}^{\mp\mathrm{i}\pi/2}\mathrm{e}^{-\rho}\mathrm{e}^{\mp\mathrm{i}\nu\pi/2}\left(\frac{\cos^2\theta}{\sin\theta}\mathrm{e}^{\pm\mathrm{i}\nu\pi/2}I_\nu(\rho) + \mathrm{e}^{\pm\mathrm{i}\nu\pi/2}\rho I'_\nu(\rho)\right)\right]$$
$$= \frac{z}{\pi}\sin\left(\frac{\pi z}{2}\right)\int_0^\infty \mathrm{d}\rho \ \rho^{-z-1}\ln\left[\sqrt{\frac{2\pi}{\rho}}\mathrm{e}^{-\rho}\left(\rho I'_\nu(\rho) + \frac{\cos^2\theta}{\sin\theta}I_\nu(\rho)\right)\right] \tag{4.14}$$

valid for $0 < \theta \leqslant \pi/2$. From (4.14), the function $\zeta_{\theta,\nu}$ can be continued step by step in the way already shown for the other case. Although the task of writing down some *general* recurrence—for specific values of $\zeta_{\theta,\nu}$—appears now to be much more involved, we feel the method should at least be useful for the (numerical or algebraic) computation of relations between special values of this function.

5. Final remarks

When solving the Helmholtz equation in dimensions higher than $D = 1$, the radial part still contains the Bessel operator, of course, but the eigenvalues are degenerated due to the different contributions of each angular mode. This occurrence of non-radial pieces makes the study of the corresponding zeta functions even more involved. Yet, we hope that the essential properties of these general objects can be related to the ones discussed here.

As remarked in [11], physical applications of zeta-function regularization often require explicit knowledge of the relevant ζ_A function somewhere on the negative real axis. Several examples of this sort of calculation are shown in that pioneering paper, but they only include cases where the eigenvalues are polynomial functions in the summation indices. For the zeros of the Bessel functions, this is no longer the case, and we feel that, in this mathematical sense, our work has gone one step further.

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Appendix A. Explicit values of $\zeta_\nu(2n)$ and of c_n

The expressions for $\zeta_\nu(2n)$ which follow from either (1.3) or (1.4) are:

$$
\begin{aligned}
\zeta_\nu(4) &= \frac{1}{2^4(\nu+1)^2(\nu+2)} \\
\zeta_\nu(6) &= \frac{1}{2^5(\nu+1)^3(\nu+2)(\nu+3)} \\
\zeta_\nu(8) &= \frac{5\nu+11}{2^8(\nu+1)^4(\nu+2)^2(\nu+3)(\nu+4)} \\
\zeta_\nu(10) &= \frac{7\nu+19}{2^9(\nu+1)^5(\nu+2)^2(\nu+3)(\nu+4)(\nu+5)} \\
\zeta_\nu(12) &= \frac{21\nu^3+181\nu^2+513\nu+473}{2^{11}(\nu+1)^6(\nu+2)^3(\nu+3)^2(\nu+4)(\nu+5)(\nu+6)} \\
\zeta_\nu(14) &= \frac{33\nu^3+329\nu^2+1081\nu+1145}{2^{12}(\nu+1)^7(\nu+2)^3(\nu+3)^2(\nu+4)(\nu+5)(\nu+6)(\nu+7)} \\
\zeta_\nu(16) &= \frac{429\nu^5+7640\nu^4+53752\nu^3+185430\nu^2+311387\nu+202738}{2^{16}(\nu+1)^8(\nu+2)^4(\nu+3)^2(\nu+4)^2(\nu+5)(\nu+6)(\nu+7)(\nu+8)} \\
\zeta_\nu(18) &= \frac{715\nu^6+16567\nu^5+158568\nu^4+798074\nu^3+2217079\nu^2+3212847\nu+1893046}{2^{17}(\nu+1)^9(\nu+2)^4(\nu+3)^3(\nu+4)^2(\nu+5)(\nu+6)(\nu+7)(\nu+8)(\nu+9)} \\
\zeta_\nu(20) &= \frac{\begin{array}{c}(2431\nu^8+80425\nu^7+1152851\nu^6+9315667\nu^5+46240675\nu^4+143917279\nu^3 \\ +273583653\nu^2+289891557\nu+130934438)\end{array}}{2^{19}(\nu+1)^{10}(\nu+2)^5(\nu+3)^3(\nu+4)^2(\nu+5)^2(\nu+6)(\nu+7)(\nu+8)(\nu+9)(\nu+10)}. \\
&\vdots
\end{aligned}
\tag{A.1}
$$

These expressions have also appeared, under a slightly different notation, in [4]. The first results were already present in classic works such as [6].

Next, we list the values of c_n coming from (3.8)–(3.11) (which are also obtainable from (3.14)):

$$
\begin{aligned}
c_4 &= 2^{-7}(-4\nu^2+25)(4\nu^2-1) \\
c_5 &= 2^{-5}(-4\nu^2+13)(4\nu^2-1) \\
c_6 &= 2^{-10}(16\nu^4-456\nu^2+1073)(4\nu^2-1) \\
c_7 &= 2^{-5}(4\nu^2-53\nu^2+103)(4\nu^2-1) \\
c_8 &= 2^{-15}(-320\nu^6+24560\nu^4-218812\nu^2+375733)(4\nu^2-1) \\
c_9 &= 2^{-9}(-64\nu^6+2160\nu^4-15084\nu^2+23797)(4\nu^2-1) \\
c_{10} &= 2^{-18}(448\nu^6-69328\nu^4+1137428\nu^2-2215391)(4\nu^2-25)(4\nu^2-1) \\
c_{11} &= 2^{-9}(256\nu^8-17536\nu^6+290016\nu^4-1535656\nu^2+2180461)(4\nu^2-1) \\
c_{12} &= 2^{-22}\left(-21504\nu^{10}+6263040\nu^8-269700224\nu^6+3698495520\nu^4-18010382628\nu^2\right. \\
&\qquad \left.+24713030909\right)(4\nu^2-1)
\end{aligned}
$$

$$c_{13} = 2^{-11}\left(-256\nu^{10} + 31040\nu^{8} - 994000\nu^{6} + 11918092\nu^{4} - 54469646\nu^{2} + 72763141\right)(4\nu^{2}-1)$$
$$c_{14} = 2^{-25}\left(135168\nu^{12} - 64542720\nu^{10} + 4832337664\nu^{8} - 126122179840\nu^{6} + 1368164250864\nu^{4} - 5951385479128\nu^{2} + 7780757249041\right)(4\nu^{2}-1)$$
$$c_{15} = 2^{-13}\left(1024\nu^{12} - 200448\nu^{10} + 10994880\nu^{8} - 247330256\nu^{6} + 2484141552\nu^{4} - 10387464744\nu^{2} + 13342715521\right)(4\nu^{2}-1)$$
$$c_{16} = 2^{-31}\left(-1757184\nu^{12} + 1271322624\nu^{10} - 145041664768\nu^{8} + 5847088874752\nu^{6} - 99257540690736\nu^{4} + 663063181817176\nu^{2} - 1052358696484885\right)\times(4\nu^{2}-25)(4\nu^{2}-1)$$
$$c_{17} = 2^{-17}\left(-16384\nu^{14} + 4845568\nu^{12} - 418833408\nu^{10} + 15765227264\nu^{8} - 290506574528\nu^{6} + 2613597023568\nu^{4} - 10302661991788\nu^{2} + 12878188618117\right)(4\nu^{2}-1) \tag{A.2}$$

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